Resit Exam - Statistics (WBMA009-05) 2023/2024

Date and time: February 2, 2024, 15.00-17.00h
Place: Exam Hall 1, Blauwborgje 4

Rules to follow:

- This is a closed book exam. Consultation of books and notes is **not** permitted. You can use a simple (non-programmable) calculator.
- Write your name and student number onto each paper sheet. There are 3 exercises and you can reach 90 points. ALWAYS include the relevant equation(s) and/or short derivations.
- **We wish you success with the completion of the exam!**

START OF EXAM

1. **Sample from the Poisson distribution.** [50]
   Let $X_1, \ldots, X_n$ be a sample from the Poisson distribution with parameter $\lambda > 0$.

   (a) Determine a sufficient statistic for λ. [5]

   (b) Compute the Maximum Likelihood (ML) estimator of λ. [5]
   You don't need to check via the 2nd derivative whether it is really a maximum.

   (c) Compute the Fisher information $I(\lambda)$ (for a sample of size $n = 1$). [5]

   (d) Assume that the sample size is $n = 100$ and that the mean of the observations is $\bar{x}_{100} = 5$. Give an asymptotic two-sided 80% confidence interval for λ. [10]

   (e) Suppose that $n = 3$. Show that the estimator $\hat{\lambda}_\star = \frac{X_1+2X_2+3X_3}{6}$ is unbiased and check whether $\hat{\lambda}_\star$ attains the Cramer-Rao bound. [5]
   Use the result from (a) and the Rao-Blackwell theorem to derive an improved estimator $\hat{\lambda}_\diamond$ with $Var(\hat{\lambda}_\diamond) \leq Var(\hat{\lambda}_\star)$. [8]
   Does the new estimator $\hat{\lambda}_\diamond$ attain the Cramer-Rao bound? [2]

   (f) Derive the uniform most powerful (UMP) test for $H_0 : \lambda = 1$ vs. $H_1 : \lambda = 3$ to the level $\alpha = 0.05$. In your derivation let the symbol $q_{\lambda,\alpha}$ denote the α quantile of a Poisson distribution with parameter λ. [10]

<u>HINT 1</u>: A Poisson distribution with parameter $\lambda > 0$ has density

$$p(x|\lambda) = e^{-\lambda} \cdot \frac{\lambda^x}{x!} \qquad (x \in \mathbb{N}_0)$$

The expectation is λ and the variance is λ.

<u>HINT 2</u>: $Y := X_1 + \ldots + X_n$ has a Poisson distribution with parameter $n\lambda$.

<u>HINT 3</u>: For the relevant quantiles see Exercise 3.

2. **Maximum Likelihood (ML) estimator.** [20]
Let X be a random variable with sample space $S = \{0, 1, 2, 3\}$ and density

$$p(x|\theta) = \begin{cases} 2\theta/3 & \text{for } x = 0 \\ \theta/3 & \text{for } x = 1 \\ 2(1-\theta)/3 & \text{for } x = 2 \\ (1-\theta)/3 & \text{for } x = 3 \end{cases}$$

where $\theta \in [0, 1]$ is an unknown parameter.

Suppose that a sample of size $n = 10$ has been taken from such a distribution and that the ten realisations are: $(x_1, \ldots, x_{10}) = (3, 0, 2, 1, 3, 2, 1, 0, 2, 1)$.

(a) Derive and provide the maximum likelihood (ML) estimator of θ.
If applicable, check via the 2nd derivative whether it is really a maximum. [10]

(b) Check whether the ML estimator is unbiased or asymptotically unbiased. [10]

3. **Neyman Pearson (UMP test).** [20]
Consider a random sample of size $n = 9$:

$$X_1, \ldots, X_9 \sim N(\mu, \sigma^2)$$

where $\sigma^2 = 1$ is known and μ is unknown.

(a) Derive the uniform most powerful test to the level $\alpha = 0.1$ for the problem

$$H_0 : \mu = 0 \text{ vs. } H_1 : \mu = 1$$ [15]

(b) Check if the power of the test is greater than 0.95. [5]

HINT 1: Recall the density of the $N(\mu, \sigma^2)$:

$$f(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} \cdot \exp\left\{-\frac{1}{2} \cdot \frac{(x-\mu)^2}{\sigma^2}\right\} \qquad (x \in \mathbb{R})$$

HINT 2: The relevant quantiles are provided in Table 1 below.

α	0.025	0.05	0.1	0.5
q_α	-1.96	-1.64	-1.28	0

Table 1: Quantiles q_α of the $\mathcal{N}(0, 1)$ distribution.